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A MULTI-STEP METHOD FOR MATRIX CONDENSATION OF FINITE ELEMENT MODELS

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1. INTRODUCTION

Dynamic condensation, as an efficient technique for model reduction, was originally applied to large finite element models to give faster computation of the natural frequencies and corresponding mode shapes. In recent years, it has also been used in test analysis-model correlation, vibration control, structural dynamic optimization, dynamic response analysis, non-linear dynamics and so on. Using this technique, the stiffness and mass matrices of a full size modal can be condensed to the size of the experimental model and the measured mode shapes can also be expanded to the full size of the finite element model.

Since Guyan [1] and Irons [2] firstly proposed the dynamic condensation technique in 1965, many kinds of algorithms have been developed. Among them, the multi-step or iterative methods are the most efficient ones since for the iterative schemes [3–7], the dynamic condensation matrix is updated repeatedly until a desired convergent value is obtained, the selection of master and slave degrees of freedom does not have much effect on the accuracy. Unfortunately, the convergence of all the multi-step schemes of dynamic condensation proposed in the past is very slow, especially when the approximate values are close to the exact result. Another disadvantage of these methods is that a full proof of the convergence is extremely difficult and has not been proved yet [3–7].

In this paper, a multi-step method for dynamic condensation is proposed. It has much higher accuracy than all the multi-step schemes proposed in the past. A full proof of its convergence is very simple. Not only the accuracy of eigenvalues but also of the eigenvectors are considered in every iterative step. A numerical example is also presented to show the efficiency of the proposed method.

2. THE ITERATIVE SCHEME FOR THE DYNAMIC CONDENSATION MATRIX R

2.1. *The governing equation of the dynamic condensation matrix* **R**

In finite element dynamic analysis, the general eigenproblem can be written as

$$\mathbf{K\Phi} = \mathbf{M\Phi\Lambda}, \tag{1}$$

where, $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and mass matrices of size $n \times n$, respectively. $\mathbf{M}$ is assumed to be positive definite and $\mathbf{K}$ positive definite (when $\mathbf{K}$ is positive semidefinite, an eigenvalue shifting will make it positive definite). $\mathbf{\Phi}$ and $\mathbf{\Lambda}$ are the eigenvector and eigenvalue matrices of size $n \times n$, respectively. The eigenvalues in matrix $\mathbf{\Lambda}$ are arranged in ascending order.

By assuming that total degrees of freedom of the full finite element model is divided into master degrees of freedom, which will be retained in the condensed model, and slave

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degrees of freedom, which will be condensed, the numbers of the two categories are m and s, respectively. According to this division, equation (1) can be partitioned as

$$\begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{sm} & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{mm} & \mathbf{\Phi}_{ms} \\ \mathbf{\Phi}_{sm} & \mathbf{\Phi}_{ss} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{sm} & \mathbf{M}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{mm} & \mathbf{\Phi}_{ms} \\ \mathbf{\Phi}_{sm} & \mathbf{\Phi}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}_{mm} & 0 \\ 0 & \mathbf{\Lambda}_{ss} \end{bmatrix}, \tag{2}$$

where the subscripts m and s denote the parameters associated with the master and slave degrees of freedom respectively.

Given that the $(i-1)$th approximation of the dynamic condensation matrix $\mathbf{R}$ is obtained and denoted as $\mathbf{R}^{(i-1)}$ and based on the definition [3–5], it can be expressed as

$$\mathbf{R}^{(i-1)} = \mathbf{\Phi}_{sm}^{(i-1)} (\mathbf{\Phi}_{mm}^{(i-1)})^{-1}, \qquad (i = 1, 2, \ldots). \tag{3}$$

Hence, the $(i-1)$th approximation of the first m eigenvectors $\mathbf{\Phi}_m^{(i-1)}$ (the size is $n \times m$) is

$$\mathbf{\Phi}_m^{(i-1)} = \begin{bmatrix} \mathbf{\Phi}_{mm}^{(i-1)} \\ \mathbf{\Phi}_{sm}^{(i-1)} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{R}^{(i-1)} \end{bmatrix} \mathbf{\Phi}_{mm}^{(i-1)}, \tag{4}$$

where $\mathbf{I}$ is a unit matrix of size $m \times m$.

According to the subspace iteration method, a new subspace $\mathbf{X}_m^{(i)}$ is calculated with an inverse power method as [7]

$$\mathbf{X}_m^{(i)} = \mathbf{C}\mathbf{\Phi}_m^{(i-1)}, \tag{5}$$

where

$$\mathbf{C} = \mathbf{K}^{-1}\mathbf{M}. \tag{6}$$

If the iterations proceeded using $\mathbf{X}_m^{(i)}$ as the next estimation of the subspace, then the subspace would collapse to a subspace of dimension 1 only containing the lowest eigenvector. Hence, the ith approximate eigenvector matrix $\mathbf{\Phi}_m^{(i)}$ is defined by adopting the orthonormalization technique.

$$\mathbf{\Phi}_m^{(i)} = \begin{bmatrix} \mathbf{\Phi}_{mm}^{(i)} \\ \mathbf{\Phi}_{sm}^{(i)} \end{bmatrix} = \mathbf{X}_m^{(i)} \mathbf{\Psi}^{(i)} = \begin{bmatrix} \mathbf{X}_{mm}^{(i)} \\ \mathbf{X}_{sm}^{(i)} \end{bmatrix} \mathbf{\Psi}^{(i)}, \tag{7}$$

where the eigenvector matrix $\mathbf{\Psi}^{(i)}$ is defined as

$$\mathbf{K}_{\mathbf{R}}^{(i)} \mathbf{\Psi}^{(i)} = \mathbf{M}_{\mathbf{R}}^{(i)} \mathbf{\Psi}^{(i)} \mathbf{\Omega}^{(i)}, \tag{8}$$

$$\mathbf{K}_{\mathbf{R}}^{(i)} = (\mathbf{X}_m^{(i)})^{\mathrm{T}} \mathbf{K} \mathbf{X}_m^{(i)}, \qquad \mathbf{M}_{\mathbf{R}}^{(i)} = (\mathbf{X}_m^{(i)})^{\mathrm{T}} \mathbf{M} \mathbf{X}_m^{(i)}. \tag{9}$$

Based on the definition of equation (3), the ith approximation of dynamic condensation matrix $\mathbf{R}$ is obtained

$$\mathbf{R}^{(i)} = \mathbf{\Phi}_{sm}^{(i)} (\mathbf{\Phi}_{mm}^{(i)})^{-1}. \tag{10}$$

By considering equation (7), equation (10) can be rewritten as

$$\mathbf{R}^{(i)} = \mathbf{X}_{sm}^{(i)} (\mathbf{X}_{mm}^{(i)})^{-1}. \tag{11}$$

Equation (11) tells one that the orthonormalization has no effect on the dynamic condensation matrix $\mathbf{R}$.

Since the orthonormalization procedure, which is also called the Rayleigh–Ritz procedure, is usually time-consuming, the solution time in the subspace iteration method

rises rapidly as the dimension of the subspace under consideration increases [8, 9]. This makes the proposed method more computationally efficient.

According to the partition form of equation (2) and by considering equation (4), equation (5) can be rewritten as

$$\begin{bmatrix} \mathbf{X}_{mm}^{(i)} \\ \mathbf{X}_{sm}^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{mm} & \mathbf{C}_{ms} \\ \mathbf{C}_{sm} & \mathbf{C}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{R}^{(i-1)} \end{bmatrix} \mathbf{\Phi}_{mm}^{(i-1)}. \tag{12}$$

Equation (12) is equivalent to the following two equations

$$\mathbf{X}_{mm}^{(i)} = (\mathbf{C}_{mm} + \mathbf{C}_{ms}\mathbf{R}^{(i-1)})\mathbf{\Phi}_{mm}^{(i-1)}, \qquad \mathbf{X}_{sm}^{(i)} = (\mathbf{C}_{sm} + \mathbf{C}_{ss}\mathbf{R}^{(i-1)})\mathbf{\Phi}_{mm}^{(i-1)}. \tag{13a, 13b}$$

Substituting equation (13) into equation (11), one obtains

$$\mathbf{R}^{(i)} = (\mathbf{C}_{sm} + \mathbf{C}_{ss}\mathbf{R}^{(i-1)})(\mathbf{C}_{mm} + \mathbf{C}_{ms}\mathbf{R}^{(i-1)})^{-1}. \tag{14}$$

Equation (14) has defined the ith approximation of dynamic condensation matrix. Assuming $\mathbf{R}^{(-1)}$ to be a zero matrix and substituting it into the right side of equation (14), an initial approximate matrix of dynamic condensation is obtained

$$\mathbf{R}^{(0)} = \mathbf{C}_{sm}\mathbf{C}_{mm}^{-1}. \tag{15}$$

Equation (14) and (15) are the proposed governing equations of the dynamic condensation matrix $\mathbf{R}$. There is no parameter of the condensed system in the equations. Hence it is unnecessary to calculate them in every iterative step.

2.2. *The iterative scheme*

The procedure to calculate the dynamic condensation matrix $\mathbf{R}$ by using the proposed iterative equation [14] is described as follows.

Step 1. Formulate the matrix $\mathbf{C}$ and its submatrix $\mathbf{C}_{mm}$, $\mathbf{C}_{ms}$, $\mathbf{C}_{sm}$ and $\mathbf{C}_{ss}$

$$\mathbf{C} = \mathbf{K}^{-1}\mathbf{M}, \qquad \mathbf{C} = \begin{bmatrix} \mathbf{C}_{mm} & \mathbf{C}_{ms} \\ \mathbf{C}_{sm} & \mathbf{C}_{ss} \end{bmatrix}. \tag{16a}$$

Step 2. Calculate the initial approximation of dynamic condensation matrix $\mathbf{R}$

$$\mathbf{R}^{(0)} = \mathbf{C}_{sm}\mathbf{C}_{mm}^{-1}. \tag{16b}$$

Step 3. For $i = 0, k, 2k, 3k, \ldots,$ begin the iteration:

3.1. Calculate the $i+k$th approximate dynamic condensation matrix $\mathbf{R}^{(i+k)}$ by iterating the following equation k times

$$\mathbf{R}^{(i+1)} = (\mathbf{C}_{sm} + \mathbf{C}_{ss}\mathbf{R}^{(i)})(\mathbf{C}_{mm} + \mathbf{C}_{ms}\mathbf{R}^{(i)})^{-1}. \tag{16c}$$

3.2. Calculate the condensed stiffness and mass matrices [5]

$$\mathbf{K}_{\mathbf{R}}^{(i+k)} = \mathbf{K}_{mm} + (\mathbf{R}^{(i+k)})^{\mathrm{T}}\mathbf{K}_{sm} + \mathbf{K}_{ms}\mathbf{R}^{(i+k)} + (\mathbf{R}^{(i+k)})^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{R}^{(i+k)}, \tag{16d}$$

$$\mathbf{M}_{\mathbf{R}}^{(i+k)} = \mathbf{M}_{mm} + (\mathbf{R}^{(i+k)})^{\mathrm{T}}\mathbf{M}_{sm} + \mathbf{M}_{ms}\mathbf{R}^{(i+k)} + (\mathbf{R}^{(i+k)})^{\mathrm{T}}\mathbf{M}_{ss}\mathbf{R}^{(i+k)}. \tag{16e}$$

3.3. Solve the eigenproblem of the condensed system

$$\mathbf{K}_{\mathbf{R}}^{(i+k)}\mathbf{Q}^{(i+k)} = \mathbf{M}_{\mathbf{R}}^{(i+k)}\mathbf{Q}^{(i+k)}\mathbf{\Omega}^{(i+k)}. \tag{16f}$$

3.4. Check the convergence by using the convergent criterion equation (16g)

$$|[\lambda_j^{(i+k)} - \lambda_j^{(i)}]/\lambda_j^{(i+k)}| < \varepsilon, \qquad (j = 1, 2, \ldots, m), \tag{16g}$$

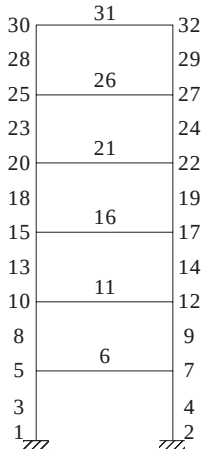

Figure 1. Schematic of a six story frame.

where $\lambda_j^{(i)}$ and $\lambda_j^{(i+k)}$ denote the ith and $(i+k)$th approximations of the jth eigenvalue and ε is an error tolerance. If the m eigenvalues have converged, exit the loop.

Step 4. Output the results: $\mathbf{R} = \mathbf{R}^{(i+k)}$, $\mathbf{K_R} = \mathbf{K_R^{(i+k)}}$, $\mathbf{M_R} = \mathbf{M_R^{(i+k)}}$ and stop.

A multi-step method for calculating the dynamic condensation matrix has been defined, based on the iterative procedure from equations (16a–g). Obviously, when the integer k is larger than one, k-1 Rayleigh–Ritz procedures are avoided. The proof of the convergence is made very simple by comparison with that of the subspace iteration method. Since the accuracy of the matrix increases quickly with the iteration, a highly accurate result can be obtained by the procedure.

3. NUMERICAL EXAMPLE

To demonstrate the efficiency of the proposed successive procedure, a six-story structure as shown in Figure 1 is considered here. The structure is modelled as a two-dimensional frame. It has a total of 32 nodes with the first two grounded. The structure has 96 degrees of freedom. For all the beams, the modulus of elasticity $= 2{\cdot}1 \times 10^{11}$ N/m^2, mass density $= 7830$ kg/m^3, area moment of inertia $= 0{\cdot}00054$ m^4, cross-sectional area $= 0{\cdot}2$ m^2. The column and level sizes in different stories are 1·0 m and 2·0 m respectively.

The first five natural frequencies obtained from the solution of the full finite element model are shown in Table 1. These values are considered as "exact" for comparison

TABLE 1

Natural frequencies of the six story frame in Figure 1

Frequency number	Natural frequency (rad/s)
1	80·653
2	260·92
3	490·82
4	770·88
5	812·14

TABLE 2

The percent errors and CCFMV values calculated with the proposed method

Iteration number	1		2		3		4		5	
	PE	CCFMV	PE	CCFMV	PE	CCFMV	PE	CCFMV	PE	CCFMV
0	0·42174	1·0000	17·852	0·96462	68·086	0·70050	134·39	0·22688	217·57	0·49352
2	0·00000	1·0000	0·00208	1·0000	0·69905	0·99978	10·826	0·92889	17·687	0·98341
4	0·00000	1·0000	0·00000	1·0000	0·00164	1·0000	0·73818	0·99571	4·4377	0·98806
6	0·00000	1·0000	0·00000	1·0000	0·00000	1·0000	0·04814	0·99970	0·62916	0·99762
8	0·00000	1·0000	0·00000	1·0000	0·00000	1·0000	0·00318	0·99998	0·07986	0·99883
10	0·00000	1·0000	0·00000	1·0000	0·00000	1·0000	0·00021	1·0000	0·01002	0·99972
12	0·00000	1·0000	0·00000	1·0000	0·00000	1·0000	0·00001	1·0000	0·00126	0·99995

TABLE 3

The percent errors and CCFMV values calculated with methods in refernces [5–7]

Iteration number	1		2		3		4		5	
	PE	CCFMV	PE	CCFMV	PE	CCFMV	PE	CCFMV	PE	CCFMV
0	0·46732	1·0000	19·113	0·96058	72·133	0·67563	145·42	0·22090	263·19	0·39268
2	0·00001	1·0000	0·06035	0·99997	2·1891	0·99716	13·640	0·89677	17·971	0·98106
4	0·00001	1·0000	0·03491	0·99999	1·0912	0·99862	3·6491	0·97270	6·5767	0·97634
6	0·00000	1·0000	0·02596	0·99999	0·82139	0·99903	2·2264	0·98505	3·0658	0·98323
8	0·00000	1·0000	0·02064	0·99999	0·65581	0·99926	1·6964	0·98982	1·9944	0·99010
10	0·00000	1·0000	0·01703	0·99999	0·54287	0·99941	1·3910	0·99227	1·4652	0·99405
12	0·00000	1·0000	0·01442	0·99999	0·46078	0·99951	1·1815	0·99375	1·1387	0·99630

purposes. When condensed the horizontal displacements at nodes 20, 23, 25, 28 and 30 are selected as master degrees of freedom.

To demonstrate the convergence of the iterative procedure, a percentage error for natural frequencies and a CCFMV, which means the correlation coefficient for modal vector, for corresponding mode shapes are defined here. The percentage error is

$$\mathbf{PE}(\omega_{appr}^{(i)}) = ([\omega_{appr}^{(i)} - \omega_{exact}]/\omega_{exact}) \times 100\% \tag{17}$$

where, $\omega_{appr}^{(i)}$ denotes the ith ($i = 0, 1, \ldots$) approximation of frequency. The CCFMV is defined as

$$\mathrm{CCFMV}(\phi_j, \phi_j^{(i)}) = \phi_j^{\mathrm{T}} \cdot \phi_j^{(i)} / \{(\phi_j^{\mathrm{T}} \cdot \phi_j)[(\phi_j^{(i)})^{\mathrm{T}} \cdot \phi_j^{(i)}]\}^{1/2} \tag{18}$$

where, ϕ_j and $\phi_j^{(i)}$ are the exact and ith approximation of the jth ($j = 1, 2, \ldots, m$) eigenvector, respectively. A CCFMV value close to 1 suggests that the two modes or vectors are well correlated and a value close to 0 indicates uncorrelated modes. Comparing e_j defined in reference [3], the CCFMV value is much more convenient for examining the accuracy of the eigenvectors.

The percentage errors of the first five frequencies and the CCFMV values of these corresponding mode shapes obtained from two kinds of method are listed in Tables 2 and 3, respectively. The results obtained from the methods in reference [3, 4] are approximately equal to those in reference [5–7] and not listed in this paper.

The results in Tables 2 and 3 clearly indicate that the proposed procedure has a much higher accuracy than the schemes in references [3–7]. After 12 iterations, the largest percentage errors in Tables 2 and 3, for example, are 0·00126% and 1·1387%, respectively. The former is about one thousandth of the latter. For the CCFMV values, the former is 0·99995 while the latter is 0·99630. The results also show that for the methods in references [3–7], the convergence is very slow when the approximate values are close to the exact ones. The percentage errors of four higher frequencies in Table 3 decrease very slowly with iterations after the fourth iteration. For the proposed method, the accuracy increases almost ten times for every two iterations.

4. CONCLUSIONS

Based on the iterative procedure, a multi-step method for dynamic condensation of finite element models is derived in this paper. It has much higher accuracy than all the multi-step schemes proposed in the past, especially when the approximate values are close the the exact ones. A full proof of the convergence is made simple by comparing with that of subspace iteration method. A numerical example also demonstrates the efficiency of the proposed method.

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